KILLING TENSORS AND CONFORMAL KILLING TENSORS FROM CONFORMAL KILLING VECTORS

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Abstract.
Koutras has proposed some methods to construct reducible proper conformal Killing tensors and Killing tensors (which are, in general, irreducible) when a pair of orthogonal conformal Killing vectors exist in a given space. We give the completely general result demonstrating that this severe restriction of orthogonality is unnecessary. In addition we correct and extend some results concerning Killing tensors constructed from a single conformal Killing vector. A number of examples demonstrate how it is possible to construct a much larger class of reducible proper conformal Killing tensors and Killing tensors than permitted by the Koutras algorithms. In particular, by showing that all conformal Killing tensors are reducible in conformally flat spaces, we have a method of constructing all conformal Killing tensors and hence all the Killing tensors (which will in general be irreducible) of conformally flat spaces using their conformal Killing vectors.

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1. Introduction.

A *Killing tensor of order 2* is a symmetric tensor K_{ab} such that

$$K_{(ab;c)} = 0. \tag{1}$$

In this paper only Killing and conformal Killing tensors of order 2 will be considered so in future this qualification will be assumed tacitly. Physically the interest in Killing tensors is due to their connection with quadratic first integrals of geodesic motion and separability of classical partial differential equations [1,2,3,4,5,6].

It is straightforward to show that the metric tensor **g**, as well as all symmetrised products of any Killing vectors $\boldsymbol{\xi}_I$, and, in general, a linear combination of all of these with constant coefficients, are all Killing tensors, i.e.,

$$K_{ab} = a_0 g_{ab} + \sum_{I=1}^{N} \sum_{J=I}^{N} a_{IJ} \xi_{I(a} \xi_{|J|b)} \tag{2}$$

is a Killing tensor, where ξ_I are the Killing vectors and a_0 and a_{IJ} for $J \geq I$ are constants. Here uppercase Latin indices label the Killing vectors and thus take values in the range $1 \dots N$ where N is the number of independent Killing vectors.

Such Killing tensors are called *reducible (degenerate, redundant* or *trivial)*; all other Killing tensors are called *irreducible (non-degenerate, non-redundant* or *non-trivial)*. (Kimura [7] uses 'proper' and 'improper' to distinguish between these two classes, but we shall not, since we will use those terms in a different context.)

For N Killing vectors there are in general $N(N + 1)/2$ symmetrised products of pairs of Killing vectors, and hence $1 + N(N+1)/2$ reducible Killing tensors; but of course these need not all be linearly independent. In an n-dimensional Riemannian space there exist at most $n(n+1)/2$ linearly independent Killing vectors, and the maximum number can be attained only in spaces of constant curvature; hence by substituting $N = n(n+1)/2$ we can obtain the maximum possible number of reducible Killing tensors in an n-dimensional Riemannian space. On the other hand, it is known that the maximum number of linearly independent Killing tensors in an n-dimensional Riemannian space is $n(n + 1)^2(n + 2)/12$ and the existence of this maximum number is a necessary and sufficient condition for spaces of constant curvature [8,9,10]. So, for example, in 4-dimensional spaces of constant curvature there are a maximum of 10 linearly independent Killing vectors and hence $56(= 1+10.11/2)$ reducible Killing tensors which can be constructed from the metric and the Killing vectors; whereas we know from [8,11] that the theoretical upper limit of linearly independent Killing tensors is only $50(= 4.5^2.6/12)$. Hauser and Malhiot [12] have reconciled these numbers by showing explicitly, in spaces of constant curvature, that of the 56 possible reducible Killing tensors constructed as above, only 50 are linearly independent. So, in 4-dimensional spaces

of constant curvature, all 50 Killing tensors are reducible. (This is a special case of the more general result [9,13,14,4] that in n-dimensional spaces of constant curvature all Killing tensors are reducible.)

However, it is the existence of <u>irreducible</u> Killing tensors which interests us. From the physical point of view such tensors yield quadratic first integrals which are not simply linear combinations of products of the linear first integrals associated with the Killing vectors. There are well known examples of curved spaces which have irreducible Killing tensors; for instance, in 4-dimensional spacetime, the Kerr metric [15] has one irreducible Killing tensor [1,2].

Unfortunately, comparatively few examples of irreducible Killing tensors are known explicitly since the direct integration of (1) is not easy, even though there are now computing programs available [16,17,18 and references therein]. So it would be useful to have indirect ways of determining irreducible Killing tensors.

In this paper we shall consider an indirect method of constructing irreducible Killing tensors via conformal Killing vectors which has been proposed by Koutras [19], and also used recently by Amery and Maharaj [20]. However, in these two papers the underlying principle is not completely transparent nor are the algorithms obtained the most general; this is partly due to a distraction caused by the trace-free requirement in the definitions of conformal Killing tensors which is used in these two papers [19,20]. Also in a paper by O'Connor and Prince [21] there has been an independent related discussion, but in the narrower context of a particular metric. We shall show that the arguments in these papers can be made more general than in the original presentations; in particular, we shall show that our more general approach enables us to obtain more conformal Killing tensors and hence more irreducible Killing tensors than those which can be obtained by the algorithms in [19,20]. In addition we shall take the opportunity to collect together various results and clarify different definitions in the literature.

In Section 2 we establish the basic results, and in Section 3 we highlight some special cases of these results which are then used for applications to specific metrics in Section 5. The results in Section 3 strengthen, extend, and, in one case, correct results in the earlier papers [19,20]. In Section 4 we extend a result of Weir [14] for flat spaces to conformally flat spaces and obtain the maximum number of conformal Killing tensors, which shows that they are all reducible in conformally flat spaces. The results are summarised and further work is discussed in Section 6.

2. Definitions, Properties and Theorems.

We begin with the familiar definitions in an n-dimensional Riemannian space:

A *conformal Killing vector* $\boldsymbol{\chi}$ satisfies $\chi_{(a;b)} = \vartheta g_{ab}$.

When:

$\vartheta_{,a} \neq 0$, $\boldsymbol{\chi}$ is a *proper* conformal Killing vector.

$\vartheta = 0$, $\boldsymbol{\chi}$ is an *improper* conformal Killing vector, which is just a *Killing vector*.

$\vartheta_{,a} = 0$, $\boldsymbol{\chi}$ is a *homothetic Killing vector*.

$\vartheta_{,a} = 0$, $\vartheta \neq 0$, $\boldsymbol{\chi}$ is a *proper* homothetic Killing vector.

By analogy with the conformal Killing vector, we define:

A *conformal Killing tensor of order 2* is a symmetric tensor Q_{ab} such that

$$Q_{(ab;c)} = q_{(a}g_{bc)} \tag{3}$$

and we easily see that $q_c = \frac{1}{n+2}Q_{,c} + \frac{2}{n+2}Q^i{}_{c;i}$, where $Q = Q^i{}_i$.

When:

$q_a = 0$, the conformal Killing tensor Q_{ab} is *improper*, and is simply a *Killing tensor* as defined in (1);

$q_a \neq 0$, the conformal Killing tensor Q_{ab} will be called *proper*.

q_a is a Killing vector, Q_{ab} is said to be a *homothetic* Killing tensor. (See Prince [23] for a discussion of homothetic Killing tensors.)

The physical interest in proper conformal Killing tensors is due to the fact that, although they do not generate quadratic first integrals for geodesic motion in general, they do so for null geodesics.

It is straightforward to show that any scalar multiple of the metric tensor, as well as all symmetrised products of conformal Killing vectors, and, in general, all linear combinations of these with constant coefficients, are also conformal Killing tensors.

However, a careful consideration of the definition shows that the number of linearly independent conformal Killing tensors is not finite since, if Q_{ab} is a conformal Killing tensor, then any other tensor of the form $Q_{ab} + \lambda g_{ab}$, where λ is an arbitrary function of the coordinates is also a conformal Killing tensor. To avoid the complication of this freedom it is usual to subtract off the trace and instead work with trace-free conformal Killing tensors. A *trace-free conformal Killing tensor of order 2* is a symmetric trace-free tensor P_{ab} such that $P^i{}_i = 0$ and

$$P_{(ab;c)} = p_{(a}g_{bc)} \tag{4}$$

and we easily see that $p_c = \frac{2}{n+2}P^i{}_{c;i}$.

When $p_a = 0$, the trace-free conformal Killing tensor P_{ab} is *improper* (i.e., simply a trace-free Killing tensor), and when $p_a \neq 0$, the trace-free conformal Killing tensor P_{ab} is *proper*. There is no contradiction or ambiguity between the two definitions. If a trace-free conformal Killing tensor exists then just by adding on an arbitrary trace we obtain a conformal Killing tensor; conversely, if a conformal Killing tensor exists then just by subtracting off its trace-term we obtain a trace-free conformal Killing tensor. For a conformal Killing

tensor which is 'pure trace' [24], that is $Q_{ab} = Q g_{ab}/n$, the corresponding trace-free tensor is identically zero. We also note that, as regards physical interpretation, the trace part does not contribute to the constant of motion along the null geodesics.

Since we wish to use these properties explicitly, and also to compare with earlier results, we will state the most general result as a theorem, whose proof can be checked directly.

Theorem 1. In an n-dimensional Riemannian space, if $\boldsymbol{\chi}_1, \boldsymbol{\chi}_2, \ldots, \boldsymbol{\chi}_M$ are conformal Killing vectors with associated conformal factors $\vartheta_1, \vartheta_2, \ldots, \vartheta_M$, any symmetric tensor of the form

$$Q_{ab} = \lambda g_{ab} + \sum_{I=1}^{M} \sum_{J=I}^{M} a_{IJ} \chi_{I(a} \chi_{|J|b)} \tag{5a}$$

is a conformal Killing tensor with associated vector

$$q_a = \lambda_{,a} + \sum_{I=1}^{M} \sum_{J=I}^{M} a_{IJ} (\vartheta_I \chi_{Ja} + \vartheta_J \chi_{Ia})$$

Here a_{IJ} for $J \geq I$ are constants; uppercase Latin indices take values in the range $1 \ldots M$. The corresponding trace-free symmetric tensor of the form

$$P_{ab} = \sum_{I=1}^{M} \sum_{J=I}^{M} a_{IJ} (\chi_{I(a} \chi_{|J|b)} - \frac{1}{n} \chi_I{}^c \chi_{Jc} g_{ab}) \tag{5b}$$

is a trace-free conformal Killing tensor.

Such conformal Killing tensors will be called *reducible (degenerate, redundant* or *trivial)*; all other conformal Killing tensors will be called *irreducible (non-degenerate, non-redundant* or *non-trivial)*.[†]

Corollary 1.1. In a manifold of dimension n¿2, if $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2$ are independent conformal Killing vectors with associated conformal factors $\vartheta_1(\neq 0)$ and ϑ_2 (so that $\boldsymbol{\chi}_1$ at least is not a Killing vector) then

$$\chi_{1a} \chi_{1b} \qquad \chi_{1(a} \chi_{|2|b)}$$

are <u>proper</u> conformal Killing tensors.

[†] Some caution needs to be exercised in reading the literature; in some papers the qualification *trace-free* is not included explicitly in the name and so what is sometimes called a 'conformal Killing tensor' is in fact 'a trace-free conformal Killing tensor'. In this paper, when appropriate, we shall also retain the qualification 'trace-free' to avoid any ambiguities. It should also be noted that for *reducible conformal Killing tensors* the factor on the metric is in general non-constant, unlike for *reducible Killing tensors* where the factor on the metric must be constant; this distinction is sometimes blurred in the literature, and in some cases separate definitions are not even given.

Proof. The fact that these tensors are conformal Killing tensors follows immediately from Theorem 1. Hence it remains to show they are proper. Clearly $\chi_{1a}\chi_{1b}$ is proper since its associated vector $q_a = \vartheta_1\chi_{1a}$ is obviously non-zero. The vector $q_a = (\vartheta_1\chi_{2a} + \vartheta_2\chi_{1a})/2$ associated with $\chi_{1(a}\chi_{2|b)}$ cannot vanish since otherwise $\boldsymbol{\chi}_2 = -\vartheta_2/\vartheta_1\boldsymbol{\chi}_1$. However this is not possible since independent conformal Killing vectors cannot be collinear (for dimensions $n > 2$).

Corollary 1.2. If $\boldsymbol{\chi}_1$ is a proper homothetic Killing vector with associated conformal factor h and $\boldsymbol{\chi}_2$ is a Killing vector then $\chi_{1(a}\chi_{2|b)}$ is a homothetic Killing tensor.

The proof is immediate as the associated vector $q_a = h\chi_{2a}$ is clearly a Killing vector since h is a constant and $\boldsymbol{\chi}_2$ is a Killing vector.

It is important to note that the total number of reducible conformal Killing tensors as given in Theorem 1, will in general be greater than all those reducible conformal tensors obtained by simply taking pairs of conformal Killing vectors. Furthermore, it is clear from the corollaries above that a pair of conformal Killing vectors, with at least one proper homothetic or proper conformal, cannot combine <u>directly</u> to give a Killing tensor; however, the possibility has not been ruled out that linear combinations of such pairs as in (5a) could give directly $q_a = 0$ and hence a Killing tensor. This possibility is not likely to be common; but we shall now consider a more general possibility of finding a Killing tensor.

A conformal Killing tensor Q_{ab} for which the vector $q_a(= \frac{1}{n+1}Q_{,a} + \frac{2}{n+2}Q^i{}_{a;i})$ is a gradient vector, (i.e., $q_a = q_{,a}$), will be called a *conformal Killing tensor of gradient type*. It is clear that such a conformal Killing tensor Q_{ab} will have an *associated Killing tensor K_{ab}* given by

$$K_{ab} = Q_{ab} - qg_{ab} \qquad (6a)$$

Such a K_{ab} is defined only up to the addition of a constant multiple of the metric.

If Q_{ab} is a conformal Killing tensor of gradient type then so is $Q_{ab} + \lambda g_{ab}$ for any scalar field λ and moreover they have the same associated Killing tensor. Thus in particular if Q_{ab} is a conformal Killing tensor of gradient type then so is its trace-free part P_{ab} with the same *associated Killing tensor K_{ab}* given by

$$K_{ab} = P_{ab} - pg_{ab} \qquad (6b)$$

where $p_{,a} = \frac{2}{n+2}P^i{}_{a;i}$.

Walker and Penrose [2] pointed out this property for the Kerr metric [15], and O'Connor and Prince [21] have exploited this result for the Kimura metrics [7]. Rosquist and Uggla [22] have exploited the two dimensional version of (6b) in investigating a large class of cosmological spacetimes.

In general, it is the existence of <u>irreducible</u> conformal Killing tensors that is the most interesting physically. However, in this paper we will only consider <u>reducible proper</u> conformal

Killing tensors, in the expectation that some of these may be of gradient type and hence will yield associated Killing tensors. Again, since we wish to use these properties explicitly, and also to compare with earlier results, we will state the most general result as a theorem, whose proof is direct.

Theorem 2. Consider the most general reducible conformal Killing tensor Q_{ab} of the form (5a). When there exists a scalar q such that

$$q_{,a} = \lambda_{,a} + \sum_{I=1}^{M} \sum_{J=I}^{M} a_{IJ}(\vartheta_I \chi_{Ja} + \vartheta_J \chi_{Ia})$$

then Q_{ab} is a conformal Killing tensor of gradient type and has an associated Killing tensor K_{ab} where

$$K_{ab} = Q_{ab} - q g_{ab}.$$

As noted above, the condition of being a gradient conformal Killing tensor is unaffected by the addition of an arbitrary trace. Thus the trace-free qualification is an unnecessary complication in the search for Killing tensors associated with reducible conformal Killing tensors. We simply construct proper reducible conformal Killing tensors ignoring any considerations of trace, that is setting $\lambda = 0$ in (5a), and test to see whether they are of gradient type and hence yield a Killing tensor.

If we know all the conformal Killing vectors of a given metric, Theorem 1 gives us all the reducible conformal Killing tensors as well as the corresponding trace-free reducible conformal Killing tensors, if required. Of course if the metric admits N independent Killing vectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N$ then the linear space of all reducible conformal Killing tensors given by (5a) contains a linear subspace of reducible Killing tensors of the form (2). We can exclude these from consideration if we choose the basis of the conformal Killing vectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N, \boldsymbol{\chi}_{N+1}, \ldots \boldsymbol{\chi}_M$ where the $\boldsymbol{\xi}_I$'s are Killing vectors and we consider only reducible conformal Killing tensors of the form

$$Q_{ab} = \sum_{I=1}^{N} \sum_{J=N+1}^{M} a_{IJ} \xi_{I(a} \chi_{|J|b)} + \sum_{I=N+1}^{M} \sum_{J=I}^{M} a_{IJ} \chi_{I(a} \chi_{|J|b)} \tag{7}$$

where a_{IJ} (for $J \geq I$) are again constants.

We then test to see if any of these conformal Killing tensors are of gradient type (including possibility of $q_a = 0$) and if so, construct the associated Killing tensors. It is straightforward to check directly which of these Killing tensors are irreducible by comparison with equation (2). Therefore this is an indirect method to find examples of Killing tensors, most of which we expect to be irreducible, from conformal Killing vectors.

Although the observations in Theorems 1 and 2 are very simple, and underlie some of the algorithms given by Koutras [19] as well as the generalisations in [20] and the calculation of

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Killing tensors for the Kimura metric in [21], the argument was not presented so generally or explicitly in these papers. Moreover, in [19,20] there was no explicit definition of a _reducible conformal Killing tensor_ (or its trace-free counterpart) and so the trivial and quite widespread occurence of these tensors seems to have been overlooked. The absence of a means of identifying all reducible conformal Killing tensors meant that all associated Killing tensors could not be found. It would seem that the cause for these less than general results in [19,20] has to do with their emphasis on trace-free conformal Killing tensors. They sought trace-free conformal Killing tensors constructed as the symmetrised product of a pair of orthogonal conformal Killing vectors, i.e., $A_i B^i = 0$, (including the special case of the product of a null Killing vector with itself) so that $P_{ab} = A_{(a}B_{b)}$ was automatically trace-free. However, as we have seen a more general way to construct a trace-free conformal Killing tensor is simply to subtract off the trace from the symmetrised product of two conformal Killing vectors, i.e., $P_{ab} = A_{(a}B_{b)} - g_{ab}A^i B_i/n$.

Thus the original results of Koutras [19] on reducible conformal Killing tensors are valid without the orthogonality assumptions of his equations (2.3) and (2.9). We shall show that this more general approach enables us to obtain more reducible proper conformal Killing tensors and hence more Killing tensors than those which can obtained by the Koutras algorithms [19].

3 Simple Algorithms for Conformal Killing Tensors and Associated Killing Tensors.

For a given metric, it will be a straightforward, if long, procedure to find the most general conditions on the constants a_{IJ} which are required for the existence of a conformal Killing tensor of gradient type. However, often there will only be a very limited number of possibilities, which can easily be deduced. So we give some of these simpler common possibilities as corollaries:

Corollary 2.1. The symmetrised product $\xi_{(a}\chi_{b)}$ of a Killing vector ξ_a and a proper conformal Killing vector χ_a satisfying $\chi_{(a;b)} = \vartheta g_{ab}$ is a conformal Killing tensor of gradient type if and only if ξ_a is a hypersurface orthogonal vector given by $\xi_a = \kappa_{,a}/\vartheta$. The associated Killing tensor is $K_{ab} = \xi_{(a}\chi_{b)} - \kappa g_{ab}$.

Corollary 2.1.1. The symmetrised product $\xi_{(a}\chi_{b)}$ of a Killing vector ξ_a and a proper homothetic Killing vector χ_a satisfying $\chi_{(a;b)} = h g_{ab}$, where h is constant, is a homothetic Killing tensor of gradient type if and only if ξ_a is a gradient vector $\xi_a = \kappa_{,a}$. The associated Killing tensor is $K_{ab} = \xi_{(a}\chi_{b)} - \kappa g_{ab}/h$.

Corollary 2.2. The symmetrised product $\chi_{1(a}\chi_{2b)}$ of two different conformal Killing vectors, respectively satisfying $\chi_{1(a;b)} = \vartheta_1 g_{ab}$ and $\chi_{2(a;b)} = \vartheta_2 g_{ab}$ is a conformal Killing tensor of gradient type if and only if $\vartheta_2\chi_{1a} + \vartheta_1\chi_{2a}$ is a gradient vector given by $\vartheta_2\chi_{1a} +$

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$\vartheta_1 \chi_{2a} = \kappa_{,a}$. The associated Killing tensor is $K_{ab} = \chi_{1(a} \chi_{2b)} - \kappa g_{ab}$.

Corollary 2.2.1. The symmetrised product $\chi_{1(a} \chi_{2b)}$ of two different proper conformal Killing vectors respectively satisfying $\chi_{(1a;b)} = \vartheta_1 g_{ab}$ and $\chi_{(2a;b)} = \vartheta_2 g_{ab}$ which are each hypersurface orthogonal given by $\chi_{1a} = \beta_{,a}/\vartheta_2$ and $\chi_{2a} = \gamma_{,a}/\vartheta_1$ respectively, is a conformal Killing tensor of gradient type. The associated Killing tensor is $K_{ab} = \chi_{1(a} \chi_{2b)} - (\beta + \gamma) g_{ab}$.

Corollary 2.3. The double product $\chi_a \chi_b$ of a proper conformal Killing vector satisfying $\chi_{(a;b)} = \vartheta g_{ab}$ is a conformal Killing tensor of gradient type if and only if χ_a is a hypersurface orthogonal vector given by $\chi_a = \kappa_{,a}/\vartheta$. The associated Killing tensor is $K_{ab} = \chi_a \chi_b - 2\kappa g_{ab}$.

Corollary 2.3.1. The double product $\chi_a \chi_b$ of a proper homothetic Killing vector satisfying $\chi_{(a;b)} = h g_{ab}$, where h is constant, is a conformal Killing tensor of gradient type if and only if χ_a is a gradient vector field given by $\chi_a = \kappa_{,a}$. The associated Killing tensor is $K_{ab} = \chi_a \chi_b - 2\kappa g_{ab}/h$.

We can get results for some particular classes of spaces. From Corollary 2.1.1 we have directly,

Theorem 3. Any space which admits a proper homothetic Killing vector χ_a with homothetic constant h as well as a gradient Killing vector $\xi_{,a}$ also admits a Killing tensor $K_{ab} = \chi_{(a} \xi_{,b)} - \xi g_{ab}/h$.

The above Corollaries 2.1 and 2.3 are given by Koutras [19] as Theorems 2 and 4 respectively, while Theorem 3 above is given in [20]; but neither the most general possibility for two conformal Killing vectors as given in our Corollary 2.2, nor the most general result in Theorem 2 above, are given in [19], [20].

Exploiting Corollary 2.3 for gradient conformal Killing vectors gives,

Theorem 4. Any space which admits a conformal Killing vector field χ_a which is a gradient also admits the Killing tensor $K_{ab} = \chi_a \chi_b - \chi^2 g_{ab}$ where $\chi^2 = \chi_a \chi^a$.

Proof. As χ_a is a gradient vector, $\chi_{[a;b]} = 0$, and therefore $\chi_{a;b} = \vartheta g_{ab}$. Thus contracting with χ^a we have $\vartheta \chi_b = (\chi^2)_{,b}/2$ and the result follows from Corollary 2.3.

In [19] and [20] it is pointed out that since a geodesic homothetic Killing vector is a gradient, the result in Corollary 2.3.1 is applicable to such vectors. Another result for geodesic vectors can be obtained as follows.

Theorem 5. Any space which admits a proper non-null conformal Killing vector field χ_a which is geodesic (that is $\chi_{a;b} \chi^b = \lambda \chi_a$) also admits the Killing tensor $K_{ab} = \chi_a \chi_b - \chi^2 g_{ab}$.

Proof. To see this we contract the equation $\chi_{(a;b)} = \vartheta g_{ab}$ with $\chi^a \chi^b$ and obtain $\lambda \chi^2 = \vartheta \chi^2$. Thus as χ^a is non-null, $\lambda = \vartheta$. Now contracting the equation $\chi_{(a;b)} = \vartheta g_{ab}$ with χ^b and obtain $\vartheta \chi_a = (\chi^2)_{,a}/2$ and hence $K_{ab} = \chi_a \chi_b - \chi^2 g_{ab}$ is a Killing tensor.

This theorem generalises Theorem 3 of Koutras which was proved in [19] (and also quoted in [20]) for homothetic Killing vectors only. Our proof is also more direct and does not rely on the introduction of a particular coordinate system. In [19] and [20] it was also claimed to be true in the null case; but it is easy to see that the proofs break down in the null case and in fact the result is false as the following counter-example shows. Consider the metric

$$ds^2 = e^{2u}(2A(x, y, v)dudv + dx^2 + dy^2)$$

A straightforward calculation shows that $\chi^a = \delta_u^a$ is a null homothetic Killing vector with conformal factor $\vartheta = 1$. As χ^a is a null conformal Killing vector it is necessarily geodesic. The associated conformal Killing tensor Q_{ab} and vector q_a are given by

$$Q_{ab} = A^2 e^{4u}\delta_a^v\delta_b^v \qquad q_a = Ae^{2u}\delta_a^v$$

A simple calculation shows q_a is not a gradient vector.

Note also that for non-null vectors Theorem 4 follows from Theorem 5 as a gradient conformal Killing vector is necessarily geodesic. For the null case a gradient conformal Killing vector is necessarily a Killing vector and so in this case the associated Killing tensor is necessarily reducible.

We emphasise again that in all of these cases the associated Killing tensors may or may not be reducible; in each individual space under consideration it would be necessary to check directly whether each Killing tensor can be reduced to a linear combination of the metric and products of pairs of Killing vectors as in equation (2).

4. Conformal Transformations.

It is well known that, if χ^a is a conformal Killing vector of the metric g_{ab} with confomal factor ϑ then it is also a conformal Killing vector of the conformally related metric $\tilde{g}_{ab} = e^{2\Omega}g_{ab}$ with conformal factor $\tilde{\vartheta} = \vartheta + \Omega_{,c}\chi^c$. We now obtain the analogous result for conformal Killing tensors:

Theorem 6. If Q^{ab} is a conformal Killing tensor satisfying $\nabla^{(a}Q^{bc)} = q^{(a}g^{bc)}$, then Q^{ab} is also a conformal Killing tensor of the conformally related metric $\tilde{g}_{ab} = e^{2\Omega}g_{ab}$. Q^{ab} satisfies $\tilde{\nabla}^{(a}Q^{bc)} = \tilde{q}^{(a}\tilde{g}^{bc)}$, where $\tilde{q}^a = q^a + 2\Omega_{,d}Q^{da}$.

Proof. The proof is straightforward involving an evaluation of $\tilde{\nabla}^{(a}Q^{bc)}$ using the result that

$$\tilde{\Gamma}_{bc}^a = \Gamma_{bc}^a + \delta_b^a\Omega_{,c} + \delta_c^a\Omega_{,b} - \Omega^{,a}g_{bc}$$

We cannot determine the number of linearly independent conformal Killing tensors because of the freedom in their trace; but we can consider the number of linearly independent trace-free conformal Killing tensors. From Theorem 6 and the analogous result for conformal Killing vectors we have,

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Corollary 6.1. The number of linearly independent trace-free conformal Killing tensors is invariant under conformal change of the metric. The number of linearly independent reducible trace-free conformal Killing tensors is similarly invariant.

The maximum number of trace-free conformal Killing tensors in an n (> 2)-dimensional Riemannian space has been found by Weir [14] to be $(n-1)(n+2)(n+3)(n+4)/12$, and he has shown that this number is attained in flat space.

For M conformal Killing vectors there are in general $M(M+1)/2$ symmetrised products of pairs of conformal Killing vectors; hence, in conformally flat spaces, we can construct $M(M+1)/2$ reducible trace-free conformal Killing tensors. In an n-dimensional Riemannian space there exist at most $(n+1)(n+2)/2$ linearly independent conformal Killing vectors, and the maximum number can be attained only in conformally flat spaces. Hence by substituting $M = (n+1)(n+2)/2$ we can obtain the maximum possible number of reducible conformal Killing tensors in an n-dimensional Riemannian space; but of course these need not all be linearly independent. (For example, in 4 dimensions there are 120 reducible trace-free conformal Killing tensors which can be constructed from the metric and the conformal Killing vectors, while the theoretical upper limit of linearly independent trace-free conformal Killing tensors is only 84.) However, Weir [14] has shown explicitly, in n (> 2)-dimensional flat spaces, that of the $M(M+1)/2$ possible trace-free conformal Killing tensors constructed as above, only $(n-1)(n+2)(n+3)(n+4)/12$ are linearly independent. So, in n (> 2)-dimensional flat spaces all $(n-1)(n+2)(n+3)(n+4)/12$ trace-free conformal Killing tensors are reducible [14].[†] It should be emphasised that these results do not apply to two dimensional spaces, where Rosquist and Uggla have found some quite different results [22].

Applying Corollary 6.1 we can extend Weir's results to conformally flat spaces:

Corollary 6.2. The maximum number of linearly independent trace-free conformal Killing tensors in n (> 2) dimensions is $(n-1)(n+2)(n+3)(n+4)/12$ and is attained in conformally flat spaces. In this case all the trace-free conformal Killing tensors are reducible.

5. Examples.

The results in the earlier sections are generally valid in n dimensions. However, our main interest will be applications in 4-dimensional spacetime. For a given metric with known Killing and conformal Killing vectors, we first find all <u>proper</u> reducible conformal Killing tensors using Theorem 1, and can then easily find the trace-free versions if required. (We

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[†] Actually Weir only shows that the linear space of trace-free reducible conformal Killing tensors is spanned by a certain subset of $(n-1)(n+2)(n+3)(n+4)/12$ such tensors. However it is easy to check that the tensors in this spanning set are linearly independent.

could of course also write down all reducible Killing tensors, but we concentrate on those conformal Killing tensors given by (7) that may lead to irreducible Killing tensors.) We could then find all conformal Killing tensors of gradient type, and hence all associated Killing tensors via Theorem 2. Alternatively we can use the corollaries in Section 3 when the cases are simple.

Kimura metric.
The Kimura metric (type I in [7], and also considered in [19] and [20]) given by

$$ds^2 = \frac{r^2}{b}dt^2 - \frac{1}{r^2b^2}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,$$

is of Petrov type D with a non-zero energy momentum tensor.
There are <u>four Killing vectors</u>

$$\xi_1{}^a = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \qquad \text{and} \qquad \xi_2{}^a = \frac{\partial}{\partial\phi}$$

$$\xi_3{}^a = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi} \qquad \text{and} \qquad \xi_4{}^a = \frac{\partial}{\partial t}$$

which in covariant form are

$$\xi_{1a} = -r^2\sin\phi\,\theta_{,a} - r^2\sin\theta\cos\theta\cos\phi\,\phi_{,a} \qquad \text{and} \qquad \xi_{2a} = -r^2\sin^2\theta\,\phi_{,a}$$

$$\xi_{3a} = r^2\cos\phi\,\theta_{,a} - r^2\sin\theta\cos\theta\sin\phi\,\phi_{,a} \qquad \text{and} \qquad \xi_{4a} = \frac{r^2}{b}t_{,a}$$

The Killing vectors ξ_{1a} and ξ_{3a} are not hypersurface orthogonal, but ξ_{2a} and ξ_{4a} are (but are not gradient).
There are also <u>two proper conformal Killing vectors</u> with conformal factors r and rt respectively

$$\chi_1{}^a = r^2\frac{\partial}{\partial r} \qquad \text{and} \qquad \chi_2{}^a = r^2t\frac{\partial}{\partial r} - \frac{1}{br}\frac{\partial}{\partial t}$$

which are both gradient vectors $\chi_{1a} = -\left(r/b^2\right)_{,a}$ and $\chi_{2a} = -\left(rt/b^2\right)_{,a}$ respectively.
Reducible Proper Conformal Killing Tensors
From Theorem 1 we can immediately write down 11 reducible proper conformal Killing tensors from the symmetrised products of each proper conformal Killing vector with each Killing vector, together with the symmetrised products of the proper conformal Killing vectors $\xi_{1(a}\chi_{1b)}$, $\xi_{1(a}\chi_{2b)}$, $\xi_{2(a}\chi_{1b)}$, $\xi_{2(a}\chi_{2b)}$, $\xi_{3(a}\chi_{1b)}$, $\xi_{3(a}\chi_{2b)}$, $\xi_{4(a}\chi_{1b)}$, $\xi_{4(a}\chi_{2b)}$, $\chi_{1a}\chi_{1b}$, $\chi_{2a}\chi_{2b}$, $\chi_{1(a}\chi_{2b)}$; it is straightforward to find the trace-free versions. These 11 tensors will not necessarily be linearly independent of each other.

Killing Tensors

Although the Killing vectors ξ_{2a} and ξ_{4a} are hypersurface orthogonal, neither are compatible with the conformal factors r and rt respectively of the two proper conformal Killing vectors to enable Corollary 2.1 to be used. On the other hand χ_{1a} and χ_{2a} are gradient vectors and so by Theorem 4 we obtain respectively two Killing tensors with non-zero components,

$$K_1^{tt} = \frac{1}{b}$$

$$K_1^{\theta\theta} = -\frac{1}{b^2}$$

$$K_1^{\phi\phi} = -\frac{1}{b^2 \sin^2 \theta}$$

and

$$K_2^{tt} = b^2 + \frac{1}{r^2}$$

$$K_2^{tr} = -btr$$

$$K_2^{\theta\theta} = -t^2$$

$$K_2^{\phi\phi} = -\frac{t^2}{\sin^2 \theta}.$$

Furthermore, noting that $\vartheta_1\chi_{2a} + \vartheta_2\chi_{1a} = -r\left(\frac{rt}{b^2}\right)_{,a} - rt\left(\frac{r}{b^2}\right)_{,a} = -\left(\frac{r^2t}{b^2}\right)_{,a}$ also enables Corollary 2.2 to be applied to χ_{1a} and χ_{2a} giving the Killing tensor with non-zero components,

$$K_3^{tt} = 2\frac{t}{b}$$

$$K_3^{tr} = -\frac{r}{b}$$

$$K_3^{\theta\theta} = -2\frac{t}{b^2}$$

$$K_3^{\phi\phi} = -\frac{t^2}{\sin^2 \theta}b^2.$$

To check whether there may be more, less obvious Killing tensors, we apply Theorem 2 in the most general case and consider the vector

$$a_{55}r(r^2/b^2)_{,a} + a_{66}rt(rt/b^2)_{,a} + (a_{15}r + a_{16}rt)(-r^2 \sin \phi\, \theta_{,a} - r^2 \sin \theta \cos \theta \cos \phi\, \phi_{,a})$$
$$+ (a_{25}r + a_{26}rt)(-r^2 \sin^2 \theta\, \phi_{,a}) + (a_{35}r + a_{36}rt)(r^2 \cos \phi\, \theta_{,a} - r^2 \sin \theta \cos \theta \sin \phi\, \phi_{,a})$$
$$+ (a_{45}r + a_{46}rt)(\frac{r^2}{b} t_{,a}) + a_{56}(rt(r^2/b^2)_{,a} + r(rt/b^2)_{,a})$$

It is immediately obvious that for this vector to be gradient, all of $a_{15}, a_{25}, a_{35}, a_{16}, a_{26}, a_{36}$ must be zero. From the remainder we find the gradient condition is equivalent to

$$0 = (a_{45}\, r_{,[b} + a_{46}t\, r_{,[b})(t_{,a]})$$

and so a_{45} and a_{46} must also be zero. Therefore, the 3 Killing tensors found above are the only ones which can be obtained by this method.

A comparison of the Killing tensor $\mathbf{K}_1$ with the Killing vectors shows that it is in fact reducible since, [19]

$$K_{1ab} = \frac{1}{b}\xi_{4a}\xi_{4b} - \frac{1}{b^2}(\xi_{1a}\xi_{2b} + \xi_{2a}\xi_{2b} + \xi_{3a}\xi_{3b}).$$

The other two Killing tensors are irreducible since it is clearly impossible to obtain, using the Killing vectors and metric, those terms in K_{2ab} and K_{3ab} which are explicit functions of t. It is easy to confirm from observation that these three tensors are linearly independent of each other and of the metric.

(In Kimura's [7] original work he sought directly for irreducible Killing tensors, and found $\mathbf{K}_2$ and $\mathbf{K}_3$. Koutras [19] only found 8 reducible trace-free conformal Killing tensors and only the 2 Killing tensors $\mathbf{K}_1$ and $\mathbf{K}_2$, because he used his less general algorithms. However, O'Connor and Prince [21] obtained all three Killing tensors since they used the same more general argument as we have done.)

Bell-Szekeres metric

We now consider the Bell-Szekeres metric [25], a Petrov type I metric, which in coordinates (u, v, x, y) is defined by the line element

$$ds^2 = 2(u+v)^{(a^2-1)/2}dxdv - 2(u+v)^{1-a}dx^2 - 2(u+v)^{1+a}dy^2$$

where a is a constant; when $a = 0, \pm 3$ the space is Petrov type D and when $a = \pm 1$ the space is flat. We concentrate on curved spaces.

It admits the three Killing vectors

$$\xi_1^a = \frac{\partial}{\partial u} - \frac{\partial}{\partial v},$$
$$\xi_2^a = \frac{\partial}{\partial x},$$
$$\xi_3^a = \frac{\partial}{\partial y},$$

which are all hypersurface orthogonal

$$\xi_{1a} = (u+v)^{\frac{(a^2-1)}{2}}(v-u)_{,a}$$
$$\xi_{2a} = -2(u+v)^{(1-a)}x_{,a}$$
$$\xi_{3a} = -2(u+v)^{(a+1)}y_{,a},$$

None of these three vectors, nor any combination, can be gradients (except in flat space).

Furthermore, this metric also possesses <u>one proper homothetic Killing vector</u>, [26]

$$\chi^a = 4u\frac{\partial}{\partial u} + 4v\frac{\partial}{\partial v} + (1+a)^2 x\frac{\partial}{\partial x} + (1-a)^2 y\frac{\partial}{\partial y}$$

with conformal factor $3 + a^2$; it is easy to see that this vector is not gradient.

<u>Reducible Proper Conformal Killing tensors</u>

From Theorem 1 we can immediately write down 3 reducible proper homothetic Killing tensors from the symmetrised products of the proper homothetic Killing vector with each Killing vector; in addition we have one reducible proper conformal Killing from the double product of the homothetic Killing vector: $\xi_{1(a}\chi_{b)}$, $\xi_{2(a}\chi_{b)}$, $\xi_{3(a}\chi_{b)}$, $\chi_a\chi_b$; it is straight-forward to find the trace-free versions by subtracting off the traces. Since the proper homothetic vector is not null, nor orthogonal to any of the Killing vectors, these four tensors could not have been obtained from the Koutras algorithms.

<u>Killing Tensors</u>

The fact that neither the homothetic Killing vector nor the Killing vectors are gradient vectors means that via Corollaries 2.1.1 and 2.3.1 we can conclude that we cannot construct any Killing tensors by this method.

<u>*Beem metric*</u>

The Beem metric [27] is defined in coordinates (u, v, x, y) by the line element

$$ds^2 = e^{vx}dvdu + dx^2 + dy^2$$

and possesses <u>two Killing vectors</u>

$$\xi_1^a = \frac{\partial}{\partial u}$$

$$\xi_2^1 = \frac{\partial}{\partial y}$$

of which ξ_{1a} is hypersurface orthogonal

$$\xi_{1a} = \frac{1}{2}e^{vx}v_{,a}$$

and ξ_{2a} is gradient

$$\xi_{2a} = y_{,a}.$$

Furthermore, this metric admits <u>one proper homothetic vector</u>

$$\chi^a = 3u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

with homothetic factor 1; this vector is not a gradient.

Reducible Proper Conformal Killing tensors

We can construct two proper homothetic Killing tensors and one proper conformal Killing tensor by using χ_a with ξ_{1a} and ξ_{2a} respectively, as well as $\chi_a\chi_b$. Again these could not have been obtained using the Koutras algorithms.

Killing tensors

The homothetic vector together with a gradient Killing vector can be used to construct a Killing tensor according to Corollary 2:1:1, and we obtain

$$K^{uv} = -2ye^{-vx}$$
$$K^{uy} = \frac{3}{2}u$$
$$K^{vy} = -\frac{1}{2}v$$
$$K^{xy} = \frac{1}{2}x$$
$$K^{xx} = -y$$

Checking with the Killing vectors and metric we note that this Killing tensor is irreducible. We cannot obtain any more Killing tensors from Theorem 2.

Fluids with gradient conformal Killing vectors

In [30] examples of perfect and other fluid spacetimes which admit a gradient conformal Killing vector are found; for all such spaces Killing tensors can be constructed using Theorem 4. In [20] the possibility of such a construction was pointed out — but only for any of these spaces whose gradient conformal Killing vector is homothetic.

Conformally flat spaces

Conformally flat spacetimes necessarily admit 15 independent conformal Killing vectors from which 84 independent reducible conformal Killing tensors can be constructed. Hence in such spacetimes there is a rich supply of 'candidate' conformal Killing tensors which may satisfy the gradient condition and so be associated with possibly irreducible Killing tensors. The large number of candidate tensors means that a direct approach by hand calculation would be lengthy and error-prone. However the calculations involved, though lengthy, are routine and this enables them to be automated by using of a computer algebra package such as Reduce. Work is in progress on investigating a number of conformally flat spacetimes including the perfect fluid solutions [28] and the pure radiation solutions [29], the Robertson-Walker metrics and the interior Schwarzschild solution; these results will be presented elsewhere. In this paper we will restrict ourselves to a few preliminary remarks. The generic perfect fluid solutions [28] and the pure radiation solutions [29] admit no Killing vectors and so if any gradient conformal Killing tensors are found then the associated

Killing tensors will necessarily be irreducible (unless they are simply constant multiples of the metric).

Amery and Maharaj [20] found a number of conformal Killing tensors and Killing tensors in Robertson-Walker spacetimes using Koutras' algorithms, but because they used only mutually orthogonal conformal Killing vectors in their construction, they were only able to construct 39 'candidate' conformal Killing tensors. However, the Robertson-Walker metrics, being conformally flat, admit the maximal number, namely 84, of reducible conformal Killing tensors and so Amery and Maharaj's results are incomplete.

A generic Robertson-Walker metric admits 6 independent Killing vectors and so 22 (= 1 + 6.7/2) reducible Killing tensors can be constructed from the metric and the Killing vectors — of which 21 are linearly independent. Similarly for the special case of the static Einstein universe which admits a seventh Killing vector, we can construct 30 (=1+7.8/2) reducible Killing tensors from the metric and the Killing vectors — of which 27 are linearly independent. Hence, after finding the gradient conformal Killing tensors and their associated Killing tensors of the generic Robertson-Walker metric (or of the Einstein universe), we need to determine whether they are irreducible by checking if they are independent of these 21 (or 27) reducible Killing tensors. Again the high dimension of these linear subspaces involved and the routine nature of the calculations means that the computations can be automated by use of the computer algebra system Reduce.

6. Discussion

We have clarified the concept and definition of *reducible* conformal Killing tensors of order 2 and their trace-free counterparts; this enables us to write down immediately all the reducible conformal Killing tensors in a space where the conformal Killing vectors are known. By identifying those reducible conformal Killing tensors of gradient type we are able to construct associated Killing tensors, most of which we expect to be irreducible. For conformally flat spaces we have shown that all conformal Killing vectors are reducible and so they can all (including both reducible and irreducible Killing tensors) be found by this indirect method.

Of course, in more general curved spaces there are important examples of irreducible trace-free conformal Killing tensors, and also of irreducible Killing tensors which are not associated with conformal Killing vectors; such tensors cannot be obtained by the indirect method in this paper. However there are other possibilities of indirect methods which may enable us to find some of these other Killing tensors. For example, we can construct Killing tensors from Killing-Yano tensors, and conformal Killing tensors from conformal Killing-Yano tensors; we could check which of these are reducible in the sense of the respective definitions given in this paper. Moreover, any proper conformal Killing tensors which are of gradient type would then give associated Killing tensors, which in general would not be

constructed from Killing vectors, or even from Killing-Yano tensors.

References

1. B. Carter, *Phys. Rev.*, **174**, 1559 (1968).

2. M. Walker and R. Penrose, *Commun. Math. Phys.*, **18**, 265 (1970).

3. N.H.J. Woodhouse, *Commun. Math. Phys.*, **44**, 9 (1975).

4. S. Benenti, *J. Math. Phys.*, **38**, 6578 (1997).

5. S. Benenti and M. Francaviglia, *General Relativity and Gravitation Vol.1,* p. 393. Ed. A. Held, Plenum Press, New York (1979).

6. E. Kalnins and W. Miller, *SIAM J. Anal.*, **11**, 1011 (1980).

7. M. Kimura, *Tensor N.S.*, **30**, 27 (1976).

8. T.Y. Thomas, *Proc. N. A. S.*, **32**, 10 (1946).

9. G. H. Katzin and I. Levine, *Tensor N.S.*, **16**, 97 (1965).

10. C.D. Collinson, J. Phys. A: Gen. Phys., **4**, 756 (1971).

11. P. Sommers *J. Math. Phys.*, **14**, 787 (1973).

12. I. Hauser and R.J. Malhiot, *J. Math. Phys.*, **16**, 1625 (1975).

13. G. Thompson, *J. Math. Phys.*, **27**, 2693 (1986).

14. G. J. Weir, *J. Math. Phys.*, **18**, 1782 (1977).

15. R. P. Kerr, *Phys. Rev. Lett.*, **11**, 237 (1963).

16. G.C. Joly, *Gen. Rel. Grav.*, **19**, 841 (1987).

17. A. Koutras and J.E.F. Skea, *Computer Physics Communications*, **115**, 350 (1998).

18. T. Wolf, *Gen. Rel. Grav.*, **30** 124 (1998).

19. A. Koutras, *Class. Quantum Grav.*, **9**, 1573 (1992).

20. G. Amery and S.D Maharaj, *Int. J. Mod. Phys. D*, **11**, 337-351 (2002).

21. J.E.R. O'Connor and G.E. Prince, *Class. Quantum Grav.* **16**, 2885 (1999).

22. K. Rosquist and C. Uggla, *J. Math. Phys.*, **32**, 3412 (1991).

23. G. Prince, *Physics Letters*, **97A**, 133 (1983).

24. R. Geroch *Commun. Math. Phys.*, **13**, 180 (1969).

25. P. Bell and P. Szekeres, *Int. J. Theor. Phys.*, **6**, 111, (1972).

26. S.B. Edgar and G. Ludwig, *Gen. Rel. Grav.*, **34**, 807 (2002).

27. J. K. Beem, *Letters Math. Phys.*, **2**, 317 (1978).

28. H. Stephani, *Commun Math. Phys.*, **4**, 137 (1967).

29. G. Ludwig and S. B. Edgar, *Class. Quantum Grav.*, **14**, L47 (1997).

30. V. Daftardar and N. Dadhich, *Gen. Rel. Grav.*, **26**, 859 (1994).